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&

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“Saving the Earth, One Combinatorial Proof at a Time”

This project did not begin with a topic, instead, it began with a goal that developed into a series of topics that we needed a moderate understanding of in order to successfully achieve this goal. Our aim was to minimize the carbon footprint produced when Holy Cross hosts a summit. The emissions we want to minimize are the ones that produced the transit of all the visiting professors. That the professors all travel here individually makes the total miles traveled by all vehicles considerably higher, thus increasing the total emissions produced. The plan proposed is to send one van from Holy Cross to loop through all nearby colleges and pick up all of the professors so not every professor is taking their own car. This hopefully will minimize the total emissions.

The challenge of this plan can be aided or solved by some mathematical modeling, more specifically combinatorial graph theory. We began by looking at the optimal vehicle to look at the vehicle's capacity (how many passengers it can hold) as well as the emissions it produces per mile. After we found the optimal vehicle to carry the professors for our model, we applied certain facets of Hamilton cycles to minimize the distance traveled in the route that picks up the professors and takes them to Holy Cross. This brings us to the next problem, the traveling salesperson problem, which looks for a way to reach every vertex on the graph (or every stop on the trip) and go back to the original point of departure in the most efficient manner possible. Because this problem does not have a model that solves every scenario, an approximation is necessary. This is due to how time-consuming it would be to sort through every Hamilton cycle possible. After we find the optimal route, we then must devise a digraph, which is a mathematical model that will allow us to compare the seemingly best routes with the emissions produced if every professor travels on their own. That way, through our weighted digraph, we can account for all of the things that may affect the emissions, thus giving us the most accurate emissions of each route. With this method, we can compare all possible ways of traveling to Holy Cross to better discern what is the most environmentally friendly way to get there. Therefore this essay will utilize, the emissions that each vehicle produces per mile, Hamilton cycles, traveling salesperson problems, approximations to traveling salesperson problems, and finally digraphs, to achieve our goal. Once the fundamentals of each step are discussed we will apply this method to a tangible, real-world example, to provide context to our method as well as demonstrate how effective this method is.

To begin, we must find the most suitable vehicle for our goals, one that can carry enough people to make a stop at each college or university, while producing the least emissions. Looking at reporting done by the European Union, we converted their emissions per kilometer to emissions per mile and found the following: an SUV (4 people) uses 88.6 g CO_2 per mile, a small car (4 people) uses 67.6 CO_2 per mile, a bus (12.7 people) uses CO_2 109.5 per mile and finally a van (7 people) uses 92.0 CO_2 per mile[1]. Therefore we were able to discern that the van is the best for our model, because it holds the perfect number of people for our journey, and has fewer emissions per person than both the SUV and the small car. One thing to note is that the van data was estimated based on relative size and emissions to the SUV and car, however, it was not reported by the European Union. We used it because we were searching for a vehicle that holds 7 people, but one that is more efficient per mile than the bus since we were not using the maximum capacity of the bus. Although this data might be slightly erroneous based on our estimations, the main goal of the project is to demonstrate the difference between emissions between each

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[1] https://www.eea.europa.eu/media/infographics/co2-emissions-from-passenger-transport/view

professor traveling on their own and our method of picking them up my van, which will be very clear with the data we provided.

Now that we have our optimal vehicle, it is necessary to define what a Hamilton cycle is, as well as, demonstrate why it is so imperative to our modeling. Hamilton cycles were first introduced in the nineteenth century by Sir William Rowan Hamilton. As noted by the textbook, "A Hamilton cycle is a graph *G* of order *n* with a cycle of *G* of length *n.* Hence, a Hamilton cycle in the graph *G* of order *n* is a cycle *x1 - x2 - x3 - … - xn - x1* of length *n,* where *x1, x2, x3, …, xn,* are the *n* vertices of *G* in some order" [2](pg 414 - 415). In order to complete a Hamilton cycle, every vertex must be hit but not every edge has to be hit. The cycle also has to end up at the vertex in which it started. One note is that any complete graph, *G*, where the number of vertices, $n > 2$ has a Hamilton cycle. One excerpt from the book that demonstrates how many Hamilton cycles are possible in a complete graph states, "a complete graph Kn of order n > 3 has a Hamilton cycle. In fact, since each pair of distinct vertices of Kn forms an edge, each permutation of the n vertices of Kn is a Hamilton path. Since the first vertex and last vertex are joined by an edge, each Hamilton path can be extended to a Hamilton cycle. We thus see that Kn has n! Hamilton paths and (n - 1)! Hamilton cycles (corresponding to circular permutations of length n)"[3] (pg. 415). Therefore, it is clear that the number of possible cycles grows exponentially with the addition of every new vertex.

Now that a Hamilton cycle is defined, we must demonstrate its utility with respect to our goal of reducing the carbon footprint of a route that picks up professors from all relatively close colleges or universities. Because we can choose to go from any college to another, the graph of

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[2] Brualdi, Richard A. *Introductory Combinatorics*. Pearson, 2018. pg 415
[3] Brualdi, Richard A. *Introductory Combinatorics*. Pearson, 2018. pg 415

all the routes from each vertex (college) to another is a complete graph, so as a result we have ((the number of colleges) - 1)! different routes possible. Because you can reverse each path, and obtain the same total distance, there are (((the number of colleges) - 1)! / 2) unique routes. As you can see from the emissions paragraph, the emissions are per mile, so the goal of the Hamilton cycle is to minimize the mileage of the trip, thus minimizing the emissions produced on the route. The concept of finding the most efficient route has been thought of before, and it is better known as the traveling salesperson problem

The traveling salesperson problem can be shown by the example following example. A person is looking to make a trip to n number of cities in the world which includes his/her starting city. The goal of the salesperson is to make it to every city once without visiting the same city a second time. Thus, he/she would be following a Hamilton cycle G for the n cities. The problem is that because the distance between each city varies and the salesperson wants to travel the least distance possible. There is no working algorithm that works for every instance, so we must use approximations and algorithms to narrow the number of cycles to sift through. This problem was originally posed by Hassler Whitney and searches for the route that backtracks the least while also hitting every single vertex once[4]. Finding the most optimal route takes so long that google maps route optimization only allows up to 10 stops, demonstrating how necessary it is to approximate when finding the best route. In an experiment of "Where's Waldo?" where 68 points were made that represented the location of Waldo on two pages, it would take 9.53×10^{77} years for the 10 largest supercomputers to compute all possible combinations to find the best route to find Waldo. This would be equivalent to 6.35×10^{67} times longer than the universe has

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4 https://www.jstor.org/stable/pdf/167517.pdf?ab_segments=0%252Fbasic_SYC-4802%252Ftest2&refreqid=excelsior%3A6e324ea30b7d31cec1c215dee6a03865

existed[5]. Essentially, to use the traveling salesperson problem in our work, we will find a handful of routes that seem most optimal, eliminate the most illogical ones, and alter the most optimal ones until we find something slightly better than the current solution.[6]

Due to its tedious nature and how time-consuming it is, the traveling salesperson problem has many approximations to it. For our example (shown later), we found it illogical to sift through the 120 different Hamilton circles and calculate the total distances for each one, so instead we used an approximation known as the genetic algorithm, one where, "Instead of exhaustively looking at every possible solution, genetic algorithms start with a handful of random solutions and continually tinkers with these solutions — always trying something slightly different from the current solutions and keeping the best ones — until they can't find a better solution any more."[7] To be thorough, we approximated the three solutions that are the most direct, or used the least distance possible. From there, we moved some of the paths to see if we could optimize this. By immediately ruling out the unreasonable seeming routes we were able to whittle down our possibilities to a much more manageable amount.

Once we found what seems to be our most optimal route, we found it necessary to devise a model to account for all of the potential factors that might affect the overall carbon footprint produced. One combinatorial model that is extremely useful for this is a weighted digraph. On a high level, these are models where each vertex represents a different factor that may either affect the what we are looking for directly or affect another vertex that ultimately affects what we are looking for. An example from an article that used digraphs to measure air pollution in an urban

5 http://www.randalolson.com/2015/02/03/heres-waldo-computing-the-optimal-search-strategy-for-finding-waldo/
6 http://www.randalolson.com/2015/03/08/computing-the-optimal-road-trip-across-the-u-s/
7 http://www.randalolson.com/2015/03/08/computing-the-optimal-road-trip-across-the-u-s/

setting is shown below[8]. The graph on the right is directly based on the table next to it, which as you can see, has estimated base values that provide context for how they determined the weight of each arc. Now that there is a visible representation of a digraph, it is necessary to define different components of it. Digraphs are similar to other combinatorial graphs such as Hamilton graphs, however, what makes them unique is that the edges have directions called arcs, and they

Variable	Abbreviation	Estimated Base Value
1. Number of passenger miles annually	Passenger Miles	9.4 billion
2. Average fuel economy per vehicle (in miles per gallon)	Fuel Economy	11.9 mpg
3. Population size	Population Size	1.60 million
4. Average cost of car (new or used)	Cost of Car	$1350
5. Average price of trip per passenger	Price of Trip	$0.85
6. Tons of emissions per day (sum of hydrocarbon, carbon monoxide, and nitrogen oxide emissions)	Emissions	1106 tons
7. Number of accidents per year	Accidents	.234 million
8. Average delay per trip	Average Delay	2 minutes
9. Total fuel consumption annually	Fuel Consumption	604 million gals. gas

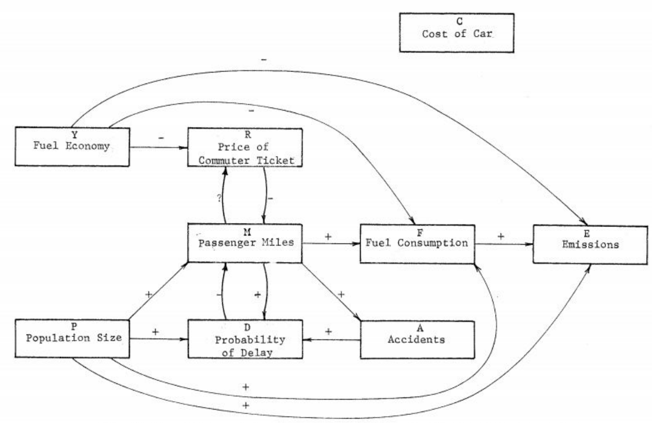


Fig. 5--Signed Digraph for Energy Use in Intraurban Commuter Transportation

Source: Refs. 8 and 9.

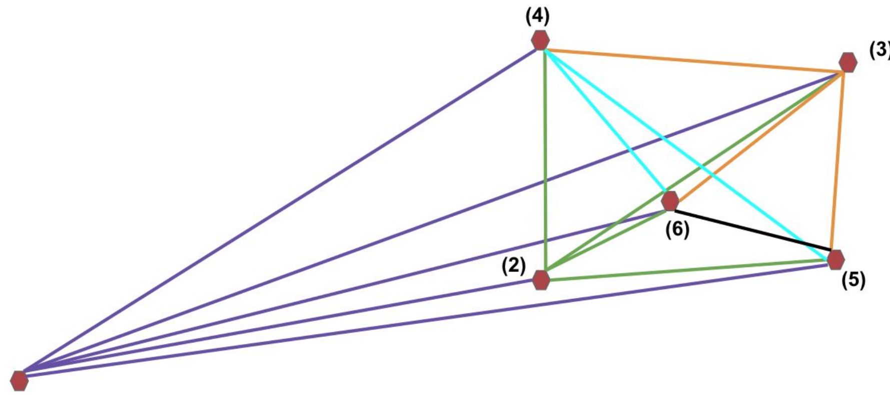
model non-symmetric relations. A digraph has the following definition, "D = (V,A) has a set of V elements called *vertices* and a set A of *ordered* pairs of not necessarily distinct vertices called *arcs*. Each arc is of the form α = (a,b), where a and b are vertices. We think of arc α as *leaving* a and *entering* b, that is pointed (or directed) from a to b."[9] For our model, we need a weighted digraph in the form a = w(s,t), where w is the weight of the arc, which demonstrates the degree of effect that the arc has on the next vertex. To make our weighted digraph there needs to exist at least two distinct vertices that have an arc connecting them with a nonnegative weight assigned to it. This is defined as the capacity per unit time. The numerical weights between arcs are

[8] https://www.rand.org/content/dam/rand/pubs/reports/2008/R1578.pdf
[9] Brualdi, Richard A. *Introductory Combinatorics*. Pearson, 2018. pg 505

determined by how each feature impacts the next one relative how it does at its estimated base value which is 1. Therefore if the vertex or component alters the total emissions in a less efficient way than the base value for the component does, its weight will be greater than 1 to increase the emissions, and if the component is more economically efficient than it would be at its base value, its capacity is less than 1 because it decreases the overall effect of emissions.

Now that we have demonstrated the emissions per vehicle, defined Hamilton cycles, introduced the traveling salesperson problem, showed its approximations, defined a digraph, provided an example of a digraph and discussed all of these models potential applications to our problem, we can finally apply it to a real-world issue. This will demonstrate how much we can reduce the carbon footprint produced when hosting an academic conference if we use the model we devised. Our simulation is as follows. Holy Cross (1) is hosting a summit on climate change and wants to invite a professor from Babson (2), Boston University (3), Bentley (4), University of Massachusetts Boston (5) and Boston College. The leaders of the summit find it hypocritical to use environmentally inefficient means of transporting each professor from their respective college to Holy Cross, so they are seeking the most optimal way to get these professors to Holy Cross while minimizing the carbon footprint produced. If you locate all of the colleges on a map

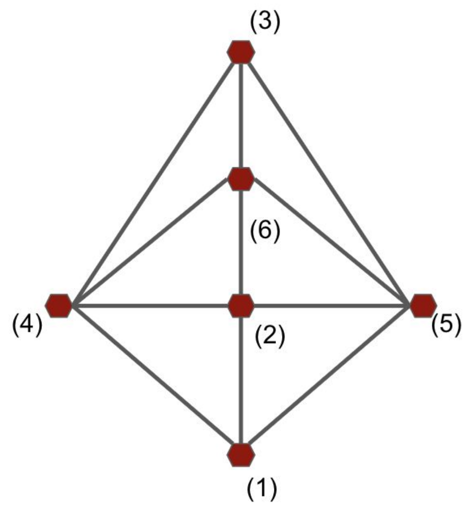
and find the most direct route between each college, you are left with a completed Hamilton graph as shown below.



(1)

Although it looked like we found straight lines between colleges, we did not. We found the fastest and most direct route between each vertex using google maps, recorded it in an excel sheet. We then depicted each path as a straight line, to better demonstrate how it can be used as a Hamilton graph. From this you can see that this graph is a complete Hamilton cycle with (n-1) routes, meaning that there are 5! or 120 different routes that you can make from the connection of these points. Of the 120 routes there are 60 unique routes, because each route can be reversed and represent another route. Just one small thing to note, this grows so exponentially, that if another college was added we would have 6! or 720 different routes (360 unique routes).

Using the genetic algorithm before we ruled out any routes that have a path directly connecting Holy Cross and Boston University, Holy Cross and Boston College, Babson and Boston University and finally Bently and University of Massachusetts Boston. This leaves us with considerably less choices. The total amount of routes, after eliminating the 4 edges, becomes 16 possible ways. We get this by rearranging the graph and moving the vertices so that it looks like:

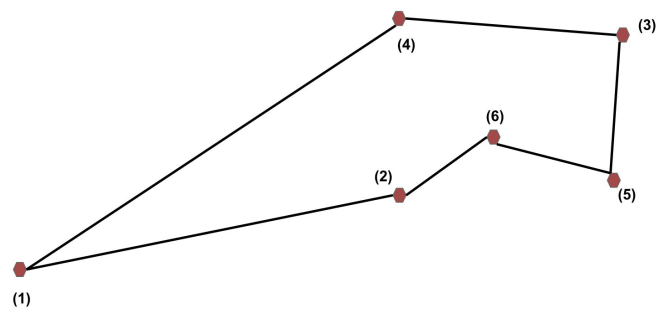


From this graph, when starting at point (1) our only options are to go to points (4), (2), or (5). If we go to point (4) we will get the same number of routes as if we were to go to point (5) and if we go to point (2) we will get one more route than by going to either (4) or (5) first. The following combinations make up the 16 routes.

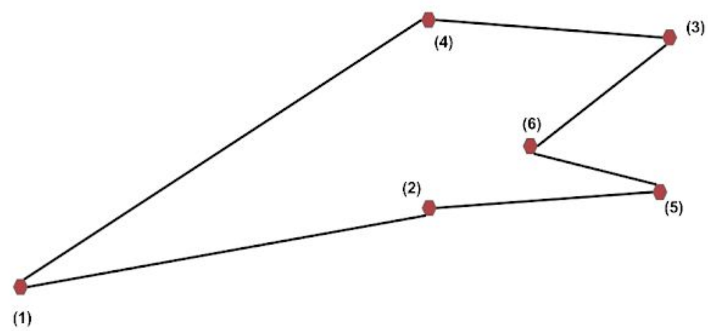
Route 1	1	5	3	6	2	4	1
Route 2	1	5	3	6	4	2	1
Route 3	1	5	3	4	6	2	1
Route 4	1	5	6	3	4	2	1
Route 5	1	5	2	6	3	4	1
Route 6	1	2	5	3	6	4	1
Route 7	1	2	5	6	3	4	1
Route 8	1	2	6	5	3	4	1
Route 9	1	2	6	4	3	5	1
Route 10	1	2	4	3	6	5	1
Route 11	1	2	4	6	3	5	1
Route 12	1	4	3	5	6	2	1
Route 13	1	4	3	6	5	2	1
Route 14	1	4	3	6	2	5	1
Route 15	1	4	6	3	5	2	1
Route 16	1	4	2	6	3	5	1

Now that we have narrowed it down to these 16 possible routes we can clearly see the three most optimal ones (or 8 unique ones).

1.)



2.)



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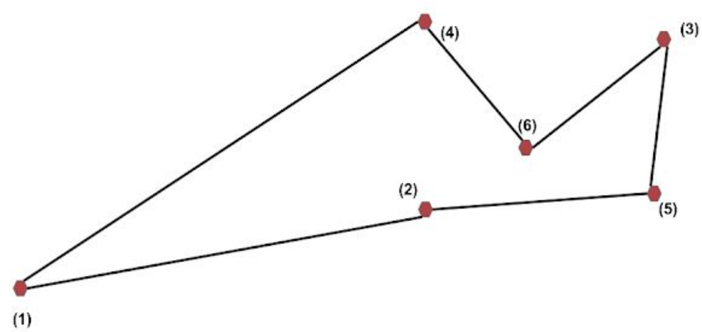
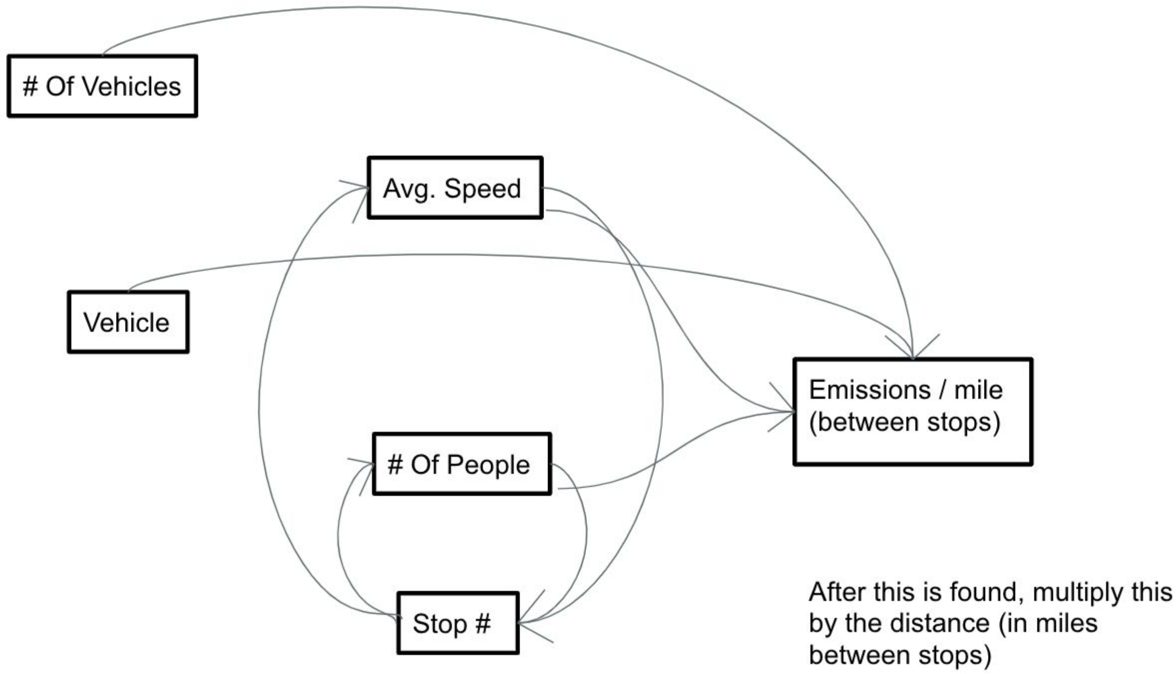


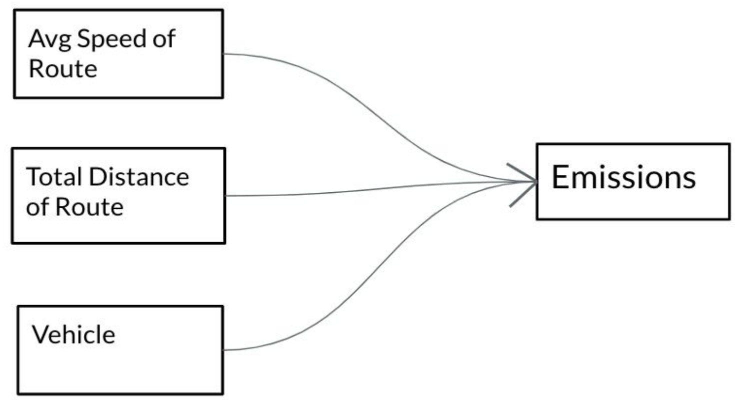
Now that we have our three optimal routes, the next step of our model is to create a digraph for the emissions produced. Let it be noted that this demonstrates 3 unique routes, and you that can do these both clockwise and counterclockwise to yield the same result. The digraph



shown below is the new digraph that we formulated. This graph accounts for the increase in emissions between each stop of the journey, because both the average speed, as well as, the number of passengers, changes between each stop. As noted by one of our sources, the optimal speed is between 55 and 65 miles per hour, therefore any slower or faster and the cars environmental efficiency will decrease[10]. As a result, the weights on the digraph weight vary from stop to stop. The type of vehicle and the number of vehicles, however, do not vary from stop to stop, they retain the same weight for the entire route. Although this digraph is very accurate, it makes calculations very tedious, because we would have to reweigh both the average speed arc and the number of passengers arc at each stop. Therefore we found it necessary to make some modifications to it so we do not have to constantly update the weights. Although it looks much more simple, our new digraph still will account for the larger components of what

[10] https://www.theguardian.com/environment/green-living-blog/2011/mar/25/hypermiling-tips

increases emissions. By removing the things that have a marginal impact on the total emissions, our new digraph uses the same weights for the entire route. It takes the average speed of the entire route and the total distance into consideration. Of course, stopping and starting will reduce the efficiency, but that is also too tedious to account for. Instead, the average speed accounts for that. Therefore this digraph will give us enough information to determine how much more efficient it is to pick up every professor by bus instead of having them travel here independently.



After tinkering with all the possible routes, using our three most optimal, and incorporating the emissions of the vehicles, we found our results. To start, if we look at our three most optimal routes, the emissions given off are 8,802.5 grams of CO2 for route one, 9,219.7 grams of CO2 for route two, and 8,654.6 grams of CO2 for route three. If the professors were to all come separately the total emissions given off, assuming that three took SUVs and two took small cars, is 11,284.1 grams of CO2. Therefore, route one would be using 78% of the emissions given off by the five cars, route two would be using 81.7%, and route three would only be using 76.7% of the total emissions the cars give off. However, even if the professors were to all take small cars, the emissions emitted would still be more than the vans. Five small cars would give off 9,511.3 grams of CO2. Even more dramatic, if every professor were to take an SUV, the emissions would be 1,2466 grams of CO2. Whereas, in comparison to the usage of the vans, there would only 69.43% of those emissions given off. To use these calculations in real terms, if there were to be two round trips made, or two conferences held where the professors would need to come to Holy Cross then go back to their respective school, then our most effective route will save a whole one-way trip as to oppose if the professors were to take three SUVs and two small cars. Our calculations and formulas can be found in the following diagram. We weighted the arc for the average speed with: weight = (base speed / avg speed) between 31.45 mph (avg speed) and 60 mph (most efficient speed) because below 31.45 mph and over 60 mph, the speed begins to make the vehicle less efficient in its carbon production per mile. Therefore the formula that we used to get total emissions is: Total Emissions = (vehicle CO_2 / mi) x (weight of avg speed) x (distance).

Individual Travelers	**Distance (mi)**	**Time (min)**	**Average Speed (mph)**	**Adjusted weight**	**Emissions**
SUVs	201	268	45	0.7	12466
Mix: 2 by car, 3 by SUV	201	268	45	0.7	11284.1
Cars	201	268	45	0.7	9511.3

Route #	**Distance (mi)**	**Time (min)**	**Average Speed (mph)**	**Adjusted weight**	**Emissions**	**Compared to 3 SUVs and 2 Cars**	
1	104	182	34.28	0.92	8802.5	78%	
2	112.6	191	35.37	0.89	9219.7	81.70%	
3	**106.9**	**180**	**35.63**	**0.88**	**8654.6**	**76.70%**	

In conclusion, by using the methods of Hamilton cycles, the traveling salesperson problem, and digraphs to find emissions from the vehicles, we have shown the benefits as to why it is impactful on the environment to use a van to pick up all of the professors instead of having them come individually, even if they are all traveling by small cars. By taking the 3 most optimal routes from the 60 unique ones that are possible, we clearly demonstrate the effectiveness of our method versus the current way of traveling. If we were given a year to do more research, we would expand our locations of schools. We would find other vehicle routes that professors would need to take and incorporate air, train, and bus travel emissions into our findings. We would also be more precise with our digraphs, incorporating the change in the number of passengers and speeds after each stop of the van. By taking these simple steps, Holy Cross as well as the colleges that come to our conferences will play their part in creating a more environmentally aware campus, and even better, a more environmentally friendly world. Our goal came from just an idea, and with that idea, we have arrived at the first step in change. It all just goes to show how we can use combinatorics today to create a better world tomorrow.

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